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Connection between Relative Gain and Control Loop Stability and Design

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The objective of this paper is to point out the relationship between the relative gain array (RGA) and control system stability and design. Rigorous stability limits are presented for a general 2 × 2 system for several cases involving both steady state and dynamic interactions. These stability limits explicitly involve the RGA. Two specific examples of interaction in distillation models are also discussed.

SCOPE

One of the most useful control tools that has appeared in the control literature is the Bristol (1966) relative gain array (RGA). The RGA has been used to assess interaction between control loops and the use of the RGA by industry has grown steadily. The RGA has the potential of giving insight into very complex control problems with a minimum of computational effort. As originally presented, the RGA involved only steady-state considerations. More recently several investigators have considered dynamic extensions of the RGA.

A legitimate question that has been raised with the traditional RGA is how can a steady-state interaction measure give an accurate indication of the dynamic behavior of control systems. The purpose of this paper is to help provide an answer to this important question. The primary focus of this paper is on control system stability although controller design is also treated. For an interacting 2 × 2 process modeled with transfer functions, rigorous relationships for control system stability which involve the RGA explicitly are presented for several cases. In addition, for two specific distillation column examples the relationship between a dynamic extension of the RGA and control system design is examined. It is shown that there is a strong connection between the RGA and control system stability and design.

CONCLUSIONS AND SIGNIFICANCE

The connection between the RGA and the stability limit for an arbitrary 2 × 2 interacting control system is established in a straightforward manner for two cases. These cases are: 1. where one loop is substantially faster than the other loop and there is a restriction on the magnitude of the RGA; and 2. where both loops are identical. The analysis of these two cases is essentially the same as that of Shinskey (1979) but more detailed. When one loop is much faster than the other, a relatively simple stability and design analysis can be carried out. When both loops are identical the denominator of the closed loop transfer functions is a quadratic which can be factored. When the RGA is constant, these two factors resemble single loop design equations. From the two factors, rigorous stability limits and frequency response designs can be carried out. For dynamic RGA's, the same two factors result and govern the stability limit of the 2 × 2 system.

The case where the two feedback loops differ and one is not substantially faster than the other is not treated in general. Rather, two specific examples involving distillation column models are treated. When a design technique proposed by Niederlinski (1971) is used, it is shown that the closed loop transfer functions can again be factored. The resulting factors involve the dynamic RGA and can be used without trial and error to assess loop stability and to design the feedback controllers.

The results presented here are significant, since they demonstrate that there is indeed a connection between control system stability and design, and the RGA. These results should also form the basis for further analysis on interacting control systems.

The RGA has received much attention in the recent literature. Applications of the RGA to distillation (Jafarey, McAvoy and Douglas, 1979; McAvoy 1979; Wang, 1979; Kominek and Smith, 1979; Jafarey and McAvoy, 1978; Shinskey, 1977a; Nisenfeld and Stravinski, 1968); heat exchanger networks (Nisenfeld, 1973); and several other processes (Shinskey, 1979) have been presented. The RGA has also been used to discuss decoupler sensitivity (McAvoy, 1979; Shinskey, 1977b). As originally presented by Bristol (1966), the RGA involved only steady-state considerations.

Recently, several investigators have considered the question of dynamic interactions. Witcher and McAvoy (1977) showed how the RGA can be extended to the frequency domain and that the extended RGA is equivalent to Rynsdorp's (1965) interaction measure. Gagnepain and Seborg (1979) extended Witcher and McAvoy's approach. Bristol (1978) has also considered dynamic extensions of the RGA and has used the concept of pinned zeroes (1980) to discuss the implications of decoupling. Lastly, Tung and Edgar (1981) used a state space approach to analyze dynamic interactions and developed a dynamic interaction measure which differs from that of Witcher and McAvoy. Tung and Edgar's interaction measure gives a ratio of open loop to closed loop effects while Witcher and McAvoy's measure gives a ratio of open loop to perfect closed loop effects.

One question that has been raised with the traditional RGA is how can a steady-state interaction measure give an accurate indication of the dynamic behavior of control systems. This question has been treated by Shinskey (1979). The main purpose of this paper is to help provide a more detailed answer to this question. By using the dynamic RGA of Witcher and McAvoy (1977), a simple analysis of a general 2 × 2 system is carried out. Several cases are treated rigorously and for these cases the relationship between control system stability and design, and the RGA is established. Also, two specific distillation models are discussed and stability limits established which involve the dynamic RGA. Controllers are designed for these two models based on open-loop frequency responses which are modified due to interaction. The controllers are designed in a straightforward manner and no trial and error is required for the two cases considered.

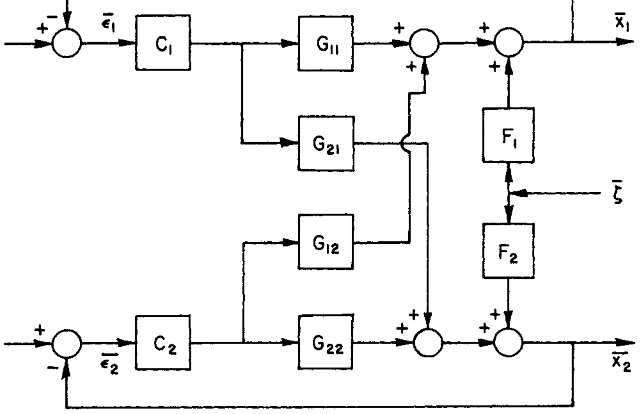

Figure 1. General 2 × 2 interacting control system studied.

Transfer Functions Considered

This paper will focus on the 2 × 2 system shown in Figure 1. The G_{ij}'s, C_i's and F_i's will be taken as transfer functions. Rijnsdorp (1965) has derived the transfer function relating the error in the upper loop, ϵ_1, to a disturbance effect ζ^* as:

$$-\frac{\bar{\epsilon}_1}{\bar{\zeta}^*} = \frac{1 + \Lambda_2}{1 + \Lambda_1 + \Lambda_2 + \Lambda_1\Lambda_2(1 - \kappa)} \tag{1}$$

where

$$\bar{\zeta}^* = \left[F_1 - F_2\frac{G_{12}}{G_{22}} \cdot \frac{\Lambda_2}{1 + \Lambda_2}\right]\bar{\zeta} \tag{2}$$

Λ_i is the product of the subloop transfer functions:

$$\Lambda_i = C_i G_{ii} \tag{3}$$

and κ is Rynsdorp's dynamic interaction measure:

$$\kappa = \frac{G_{12}G_{21}}{G_{11}G_{22}} \tag{4}$$

The variable $\bar{\zeta}^*$ does not affect loop stability or design and this is the reason for Rijnsdorp factoring it out of Eq. 1. Witcher and McAvoy (1977) have shown that for a 2 × 2 system, a dynamic relative gain element λ can be defined and this relative gain element is related to κ as:

$$\lambda = \frac{1}{1 - \kappa} = \frac{1}{1 - \dfrac{G_{12}G_{21}}{G_{11}G_{22}}} \tag{5}$$

Knowing λ allows one to construct the 2 × 2 RGA using the properties that the rows and columns sum to 1.0.

In analyzing interactions between the two loops in Figure 1, it is convenient to first consider interactions near the critical frequency of loop 1 and then consider steady-state effects. Further, for analyzing interactions near the critical frequency, three distinct cases for the speed of Λ_2 relative to Λ_1 will be treated. These cases are where the speed of Λ_2 is much faster (case 1), much slower (case 2), and equal to (case 3) the speed of Λ_1. After treating these three cases, two practical examples involving dynamic models of distillation columns will be considered.

Interactions Near Critical Frequency of Loop 1

Interactions near the critical frequency of loop 1 will be the most important in determining the stability of loop 1.

Case 1: Λ_2 Much Faster Than Λ_1. Interactions between a composition loop (Λ_1) and a flow loop (Λ_2) would be one practical example of this case.

Following Rijnsdorp (1965), Eq. 1 can be rewritten after substitution of Eq. 5 as:

$$-\frac{\bar{\epsilon}_1}{\bar{\zeta}^*} = \frac{1}{1 + \Lambda_1\left(\frac{1 + \Lambda_2/\lambda}{1 + \Lambda_2}\right)} \tag{6}$$

If C_2 is a well-designed controller with an integral mode, near the critical frequency for loop 1 the following relationship holds

$$|\Lambda_2| >> 1.0 \tag{7}$$

Equation 7 results from the fact that near the critical frequency for loop 2, $|\Lambda_2|$ will be approximately 0.5 for a reasonable gain margin. At lower frequencies, $|\Lambda_2|$ will become large due to the presence of the integral mode. It will be further assumed that near the critical frequency for loop 1 that:

$$|\Lambda_2| >> |\lambda| \tag{8}$$

Substitution of Eqs. 7 and 8 into Eq. 6 gives:

$$-\frac{\bar{\epsilon}_1}{\bar{\zeta}^*} \simeq \frac{1}{(1 + \Lambda_1/\lambda)} \tag{9}$$

Equation 9 closely resembles the traditional closed-loop transfer function for a single loop. As can be seen, the relative gain element λ appears in this transfer function. The ultimate gain for C_1 can be calculated from:

$$K_{u1I} = \frac{|\lambda|}{|G_{11}|}\bigg|_{\omega u1I} \tag{10}$$

where ω_{u1I} is calculated from:

$$\measuredangle \frac{G_{11}}{\lambda} = -\pi \tag{11}$$

Consider the case where the dynamic terms in the G_{ij}'s cancel and λ is constant, $\lambda = \lambda_o$ (the traditional relative gain situation). Assume that the gain of C_1 is set to the same fraction of its ultimate gain regardless of whether loop 2 is active or not. Then for constant λ, Eq. 10 shows that the gain of C_1 for both loops active should be λ_o times its value with loop 2 inactive. The ultimate frequency of loop 1 is not affected by interaction when λ is constant. Since values λ_o greater than 1.0 can occur, multiplying the gain of C_1 by λ_o may not be acceptable in practice. Should loop 2 fail for $\lambda_o > 2$, loop 1 would exceed its stability limit. It should be noted that Shinskey (1979) has also derived Eq. 9 and discussed its control implications. His conclusions agree with those presented above.

Case 2—Λ_2 Much Slower Than Λ_1. This case would result if Λ_1 were the flow loop of the previous case and Λ_2 were the composition loop.

Near the critical frequency of loop 1 the following relationship holds

$$|\Lambda_2| << 1.0 \tag{12}$$

Further, it will be assumed that near the critical frequency for loop 1:

$$|\Lambda_2| << |\lambda| \tag{13}$$

Substitution of Eqs. 12 and 13 into Eq. 6 gives:

$$-\frac{\bar{\epsilon}_1}{\bar{\zeta}^*} \simeq \frac{1}{1 + \Lambda_1} \tag{14}$$

Equation 14 is the traditional single loop design equation. Since λ does not appear in Eq. 14, interaction has no dynamic effect on the Λ_1 loop in this case. Cases 1 and 2 show that when one loop is significantly faster than the other loop, only the slow loop is affected by interactions near its critical frequency provided that Eqs. 8 and 13 hold.

Case 3—$\Lambda_2 = \Lambda_1$. This case would arise if one were considering two identical composition loops. For this case the denominator of Eq. 1 becomes a quadratic in Λ_1. The two roots of this quadratic are:

$$r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \lambda} \tag{15}$$

Thus, Eq. 1 can be rewritten as:

$$-\frac{\bar{\epsilon}_1}{\bar{\zeta}^*} = \frac{(\Lambda_1 + 1)}{\left(1 + \frac{\Lambda_1}{\lambda + \sqrt{\lambda^2 - \lambda}}\right)\left(1 + \frac{\Lambda_1}{\lambda - \sqrt{\lambda^2 - \lambda}}\right)} \tag{16}$$

Each of the two factors in the denominator of Eq. 16 resembles traditional control system design equations. Again consider the traditional relative gain case where $\lambda = \lambda_o$. If $\lambda_o > 1.0$, the factor with the $-\sqrt{\ }$ will always determine the stability limit. Thus to calculate the stability limit for C_1 when $\lambda_o > 1.0$, one can use the frequency response of the effective process transfer function:

$$G_{11}^* = \frac{G_{11}}{\lambda_o - \sqrt{\lambda_o^2 - \lambda_o}} \quad (\lambda_o > 1.0) \tag{17}$$

The ultimate gain of C_1 can be calculated as:

$$K_{u1I} = \frac{\lambda_o - \sqrt{\lambda_o^2 - \lambda_o}}{|G_{11}|}\bigg|_{\omega u1I} \tag{18}$$

where ω_{u1I} is calculated from:

$$\measuredangle \frac{G_{11}}{\lambda_o - \sqrt{\lambda_o^2 - \lambda_o}} = -\pi \tag{19}$$

As can be seen the relative gain element affects the stability limit. For $\lambda_o > 1.0$ the effective process gain is always greater than 1.0 and it approaches $2K_{11}$ as $\lambda_o \to \infty$. [The denominator in Eq. 17 can be written as $(\lambda_o - \lambda_o\sqrt{1 - 1/\lambda_o})$. As $\lambda_o \to \infty$ the denominator approaches 1/2 since $\sqrt{-\epsilon}$ equals $1 - \epsilon/2$.] The ultimate frequency, ω_{u1I}, is unaffected by λ_o and it is equal to ω_{u1}.

Since interaction increases the effective process gain, the controller gain must be decreased up to a maximum of 1/2 its single loop value when $\lambda_o > 1.0$.

If λ is constant and $\lambda_o < 1.0$, the factor in Eq. 14 which determines stability is the one with the $+\sqrt{\ }$. Thus, to calculate the stability limit of C_1 when $\lambda_o < 1.0$, one can use the frequency response of the effective process transfer function:

$$G_{11}^* = \frac{G_{11}}{\lambda_o + \sqrt{\lambda_o^2 - \lambda_o}} \quad (\lambda_o < 1.0) \tag{20}$$

The ultimate controller gain, K_{u1I}, and ultimate frequency, ω_{u1I}, can be calculated from Eqs. 18 and 19 with the $-\sqrt{\ }$ term replaced by a $+\sqrt{\ }$ term. The magnitude of the $1/(\lambda_o + \sqrt{\lambda_o^2 - \lambda_o})$ term is always greater than 1.0 and it contributes a constant phase lag. Thus, for $\lambda_o < 1.0$ not only is the ultimate controller gain reduced from its single loop value, but the ultimate frequency is reduced as well due to interaction.

Again it can be noted that Shinskey (1979) has treated the case where $\Lambda_1 = \Lambda_2$. His analysis is much briefer and slightly differ-

TABLE 1. EFFECT OF INTERACTION ON LOOP 1 FOR $\lambda = \lambda_o$

Speed of Λ_2 Relative to Λ_1	λ_o	Effect on Loop Stability	Response Near Steady State Relative to That for No Interaction
Much Faster	>1.0	$K_{u1I} = \lambda_o K^*_{u1}$	More Sluggish
Much Slower	>1.0	$K_{u1I} = K^*_{u1}$	More Sluggish
Equal	>1.0	$K_{u1I} = (\lambda_o - \sqrt{\lambda_o^2 - \lambda_o})K^*_{u1}$	More Sluggish
Much Faster	<1.0	$K_{u1I} = \lambda_o K^*_{u1}$	More Oscillatory
Much Slower	<1.0	$K_{u1I} = K^*_{u1}$	More Oscillatory
Equal	<1.0	$K_{u1I} = \lvert\lambda_o + \sqrt{\lambda_o^2 - \lambda_o}\rvert K^{**}_{u1}$	More Oscillatory

* $\omega_{u1I} = \omega_{u1}$,
** ω_{u1I} lower than ω_{u1} and calculated from $\measuredangle G_{11} = \measuredangle(\lambda_o + \sqrt{\lambda_o^2 - \lambda_o}) - \pi$

ent although it contains the essence of the analysis presented here. His conclusions agree with those discussed above. Shinskey also factored a quadratic in discussing the effect of interaction on controller design for $\Lambda_1 = \Lambda_2$ and limited his analysis to the case of $\lambda = \lambda_o$. Kominek and Smith (1979) also have examined a distillation control scheme where $\Lambda_1 \simeq \Lambda_2$ but they used Eq. 9 rather than Eq. 16 to carry out their analysis.

Steady-State Interactions

The traditional RGA analysis will be briefly reviewed to complete the interaction analysis of the loops shown in Figure 1. To analyze steady-state interaction effects it will be assumed that C_2 contains an integral mode. Thus, as $\omega \rightarrow 0$

$$|\Lambda_2| \rightarrow \infty \tag{21}$$

Substitution of Eq. 21 into Eq. 6, which applies in general to the loops shown in Figure 1, gives Eq. 9 provided that Eq. 8 holds. Thus, subject to the restriction given by Eq. 8, Eq. 9 holds as $\omega \rightarrow 0$ regardless of the speed of Λ_2 relative to Λ_1 and it holds near ω_{u1I} when Λ_2 is much faster than Λ_1. Near steady state λ will approach a constant value, λ_o. If $\lambda_o > 1.0$, Eq. 9 shows that the Λ_1 loop will exhibit sluggish behavior because interaction lowers the effective process and therefore loop gain. If $\lambda_o < 1.0$ the loop will be more oscillatory since the effective process gain is increased. If C_1 contains an integral mode then the same arguments given above for the Λ_1 loop could be made in reverse to discuss the steady-state effects of interaction on the Λ_2 loop.

Table 1 summarizes the results of this section for constant λ. In Table 1, K_{u1} is the ultimate controller gain for loop 1 when $\Lambda_2 = 0$ and K_{u1I} is the ultimate loop gain when Λ_2 is active. The last column in Table 1 should be interpreted with regard to how much λ_o differs from 1.0. When $\lambda_o = 1.0$, interaction does not affect the individual loop responses at all. A last point which can be made is that it is straightforward to use the previous relationships to show that when $\lambda_o < 0$ the resulting 2×2 system is unstable. For $\lambda_o < 0$ changing the pairings will result in stable control.

Equations 10, 11, 18 and 19 are particularly important. These equations give an interaction modified stability limit for the system shown in Figure 1. These equations explicitly show how λ affects loop stability. Equations 10 and 11 are valid for frequency dependent λ's (dynamic interaction) as well as for $\lambda = \lambda_o$. Similarly, if λ_o is replaced by λ in Eqs. 18 and 19, these equations hold for frequency dependent λ's. For frequency-dependent λ's, a question which arises is how to use the stability limits to design the two feedback controllers. Since the primary purpose of this paper is to demonstrate the connection between control loop stability and the RGA, this design question for frequency dependent λ's will not be treated in general. However, new results for two specific examples involving controller design for frequency dependent λ's are given below. These new results involve an extension of the previous analysis which is limited to the case where one loop is much faster than the other or where both loops are identical.

Frequency-Dependent λ's

In this section, two specific examples of frequency-dependent λ's will be examined. Both examples deal with transfer function models obtained from experimentally forcing a distillation column and then fitting the resulting data.

Example 1. Wood and Berry (1973) examined an 8 tray + reboiler distillation column separating methanol and water and published the following model for the tower

$$\begin{bmatrix} \bar{x}_D \\ \\ \bar{x}_B \end{bmatrix} = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s + 1} & \dfrac{-18.9e^{-3s}}{21s + 1} \\ \dfrac{6.6e^{-7s}}{10.9s + 1} & \dfrac{-19.4e^{-3s}}{14.4s + 1} \end{bmatrix} \begin{bmatrix} \bar{L} \\ \\ \bar{S} \end{bmatrix} \tag{22}$$

The manipulative variables were reflux and steam flow, while the controlled variables were distillate and bottoms composition. From Eqs. 5 and 22, a dynamic λ can be calculated. When the frequency response of this dynamic λ is considered, it is found that the contribution of the time constant terms is very small. Thus, for the Wood and Berry model a very accurate expression for the frequency response of λ is:

$$\lambda \simeq \frac{1}{1 - 0.502e^{-6i\omega}} \tag{23}$$

For $\omega = 0$, λ equals $\lambda_o = 2.01$, the traditional RGA element.

Example 2. Luyben and Vinante (1972) studied a 24-tray tower separating methanol and water. They presented the following transfer function model for controlling the temperatures on the 4th (stripping section) and 17th (rectifying section) trays:

$$\begin{bmatrix} \bar{T}_{17} \\ \\ \bar{T}_4 \end{bmatrix} = \begin{bmatrix} \dfrac{-2.16e^{-s}}{8.5s + 1} & \dfrac{1.26e^{-0.3s}}{7.05s + 1} \\ \dfrac{-2.75e^{-1.8s}}{8.2s + 1} & \dfrac{4.28e^{-0.35s}}{9.0s + 1} \end{bmatrix} \begin{bmatrix} \bar{L} \\ \\ \bar{S} \end{bmatrix} \tag{24}$$

When Eqs. 5 and 24 are used to calculate the frequency response of λ for this model, it is again found that the time constant terms contribute very little to the response. For this model the frequency response of λ is approximately:

$$\lambda \simeq \frac{1}{1 - 0.375e^{-0.75i\omega}} \tag{25}$$

For $\omega = 0$, Eq. 25 gives the traditional RGA value $\lambda_o = 1.6$. It is particularly interesting that these two models, derived from experimental data, produce the same functional form for the frequency response of λ.

Control System Design. In order to show how the frequency-dependent λ's influence both control system stability and design, a design technique published by Niederlinski (1971) will be used. The reason for considering Niederlinski's approach is that it can easily be analyzed by the techniques discussed here. Niederlinski has reported the successful application of his method to a number of multivariable systems. In applying

$$\frac{-\bar{\epsilon}_1}{\bar{\zeta}^*} = \frac{(1 + \Lambda_1 \alpha e^{-\tau s})}{\left(\frac{\Lambda_1}{r_1} + 1\right)\left(\frac{\Lambda_1}{r_2} + 1\right)} \quad (30)$$

where

$$r_{1,2} = \frac{-(1 + \alpha e^{-\tau s}) \pm \sqrt{(1 + \alpha e^{-\tau s})^2 - 4\alpha e^{-\tau s}/\lambda}}{(2\alpha e^{-\tau s}/\lambda)} \quad (31)$$

In Eq. 31, r_1 is the $+\sqrt{\ }$ term and r_2, the $-\sqrt{\ }$ term. In order to design the Λ_1 loop, the frequency response of the effective process transfer functions (G_{11}/r_1) and (G_{11}/r_2) can be used. These two frequency responses are shown in Figures 2 and 3 for examples 1 and 2. As can be seen, they are substantially

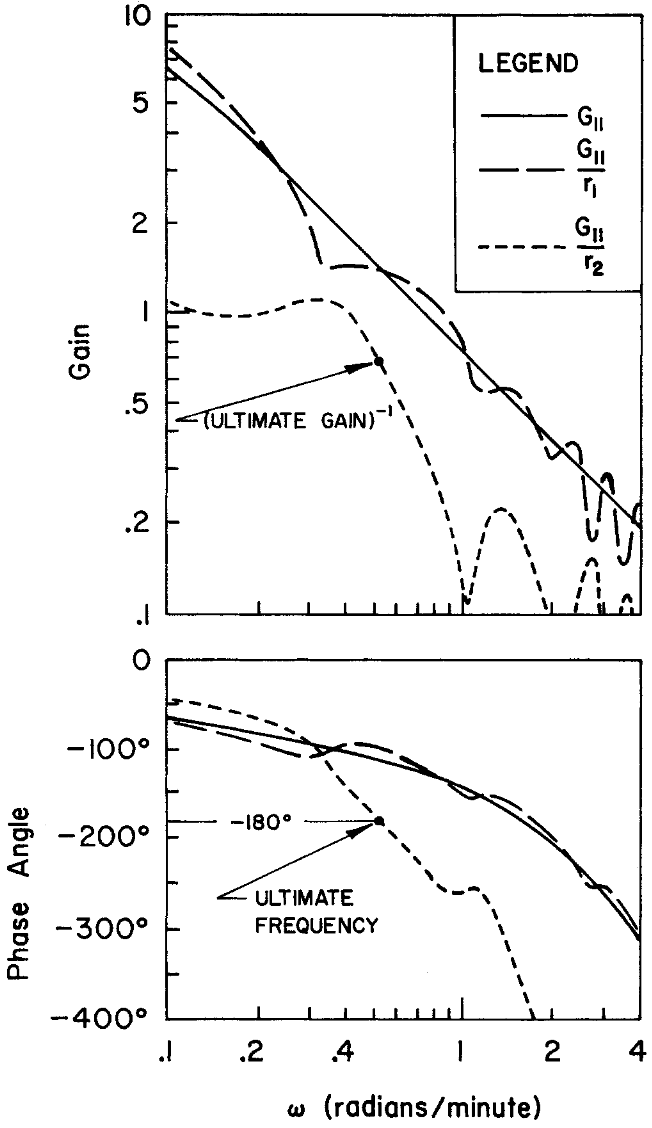

Figure 2. Frequency response for example 1.

Niederlinski's method, it will be assumed that PI controllers, which are normally used to control tower compositions, are employed. With Niederlinski's technique the same reset time is used for both controllers. Also the ratio of the controller gains is specified a priori as a constant

$$\frac{K_{C2}}{K_{C1}} = a \quad (26)$$

For the present study, a will be calculated from the ratio of the single-loop controller gains with the other loop in manual. The Ziegler Nichols (1942) rules are used to get the single loop controller gains. For example 1, $a = -0.2074$ and for example 2, $a = -1.476$. Using Eqs. 3 and 26, Λ_2/Λ_1 can be written as:

$$\frac{\Lambda_2}{\Lambda_1} = \alpha e^{-\tau s} \quad (27)$$

where

$$\alpha = a \frac{K_{22}}{K_{11}} \left(\frac{T_{11}s + 1}{T_{22}s + 1}\right) \quad (28)$$

$$\tau = \tau_{22} - \tau_{11} \quad (29)$$

For example 1, $aK_{22}/K_{11} = 0.3148$ and $\tau = 2$ and for example 2, $aK_{22}/K_{11} = 2.924$ and $\tau = -0.75$. After substituting Eqs. 5 and 27 into Eq. 1 and factoring the quadratic in the denominator, Eq. 1 can be written as

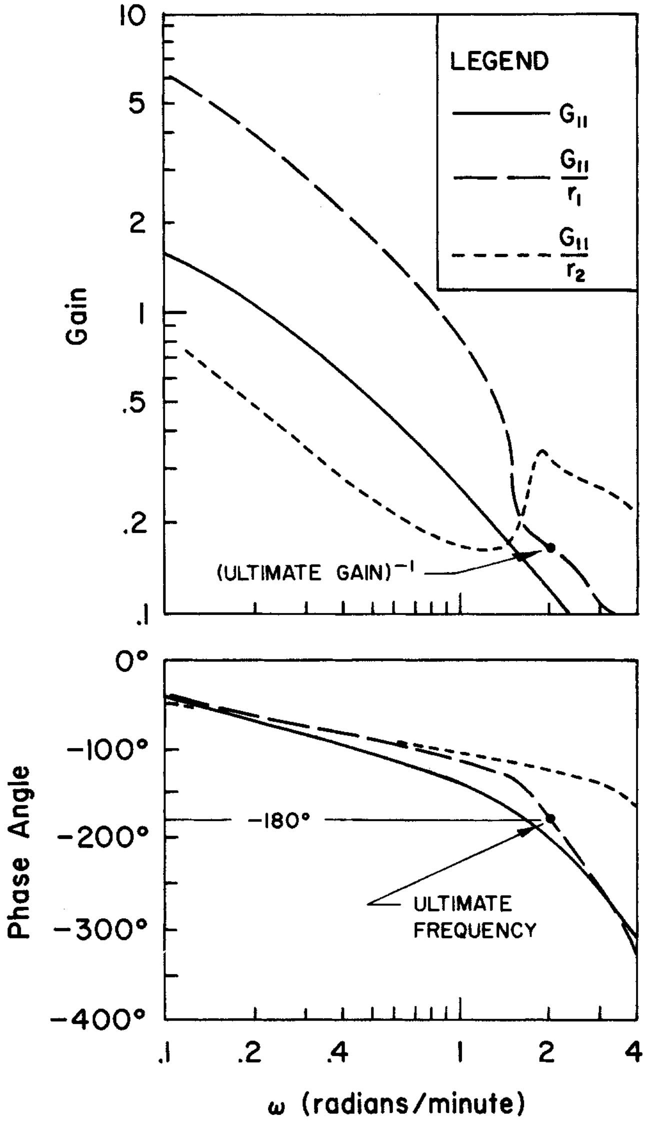

Figure 3. Frequency response for example 2.

TABLE 2. CONTROLLER PARAMETERS

Example	Loop Number	Design Method	K_C	T_R
1	1	Single Loop	0.945	3.26
1	2	Single Loop	−0.196	9.00
1	1	Interacting	0.647	10.20
1	2	Interacting	−0.134	10.20
2	1	Single Loop	−2.92	3.18
2	2	Single Loop	4.31	1.15
2	1	Interacting	−2.59	2.58
2	2	Interacting	4.39	2.58

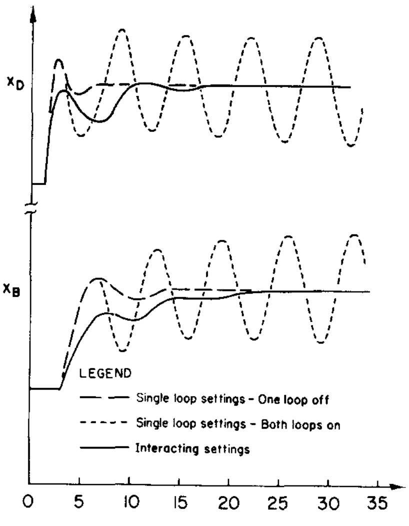

Figure 4. Set-point responses for example 1.

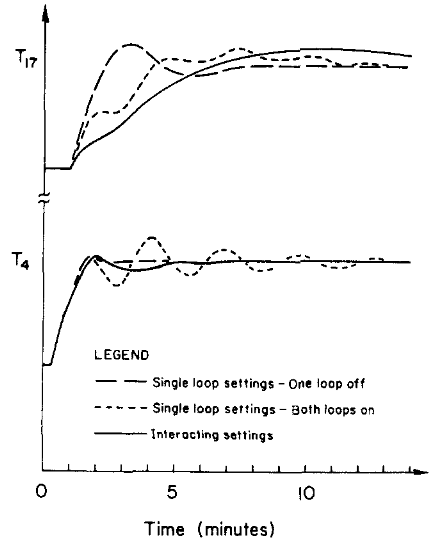

Figure 5. Set-point responses for example 2.

different from the usual frequency response obtained for single loop systems. For each example, the $-180°$ phase lag point is shown. For example 1, G_{11}/r_2 determines the ultimate frequency and stability limit, while for example 2 G_{11}/r_1 determines these two properties. Since the frequency response for both examples is reasonably well-behaved and does not show severe changes at frequencies less than the critical frequency, the Ziegler Nichols rules will be used to design C_1. After C_1 is designed, Eq. 26 and the fact that C_1 and C_2 have the same reset time can be used to design C_2. The resulting controllers are shown in Table 2 together with the corresponding single loop controllers. A point which should be emphasized here is that the above approach is straightforward and it does not require trial and error for the two examples considered.

In Figures 4 and 5, the servo responses for the two examples are shown. Only the variable whose set point is changed is shown. As can be seen, when one loop is in manual, the single-loop controllers produce a very good response. However, when both single-loop controllers are used simultaneously, the responses for example 1 are unstable and those for example 2 are highly oscillatory. The interacting controller design is more sluggish but stable. No systematic attempt was made to try to improve on the interacting designs, since the focus of this paper is on the stability question. Although the designs could probably be improved somewhat, Bhalodia and Weber (1979) reported that the optima for interacting 2×2 feedback designs is reasonably flat.

Discussion of Results

It is useful to consider the results shown in Figures 4 and 5 from the perspective of the traditional RGA. Because of its ease of calculation, the RGA is potentially an extremely important control design tool. The traditional interpretation of λ_o is that the larger it is, the more the two loops will interact. Figures 4 and 5 show this to be true, since the single loop controller settings are unstable for example 1 ($\lambda_o = 2.01$) and stable but oscillatory for example 2 ($\lambda_o = 1.60$).

One of the problems with the traditional RGA is how to use it to modify controller designs. Equations 30 and 31 and Figures 2 and 3 show how the frequency-dependent RGA enters into stability and design considerations. As Eqs. 30 and 31 and Figures 2 and 3 show the relationship of the frequency-dependent RGA to control system stability and design is complicated and several questions remain to be answered. For example, how would one design controllers for the system shown in Figure 2 if its frequency response oscillated substantially below the 180° crossover frequency? Secondly, can the traditional RGA be tied more simply to control system design? For example for the two distillation models suppose that one ignored the frequency dependence of λ and assumed $\Lambda_1 = \Lambda_2$. Table 1 then indicates that the single-loop controller gains should be lowered by $\lambda_o - \sqrt{\lambda_o^2 - \lambda_o}$. For the two distillation examples this approach produces a transient response very similar to that given by the interacting Niederlinski design. Whether or not such an approach will work in general remains to be seen. The major advantage of the approach presented here is that it will allow for further investigation and analysis of the connection between the RGA and control system stability and design.

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NOTATION

a = ratio of feedback controller gains
C_i = feedback controller transfer function for loop i
F_i = disturbance transfer function for loop i
G_{ij} = process transfer function
G_{ii*} = effective process transfer function
K_{ci} = controller gain for loop i
K_{ij} = gain of G_{ij}
K_{ui} = single-loop ultimate gain for loop i
K_{u1I} = interacting ultimate gain for loop 1
L = reflux flow
m_i = manipulated variable for loop i
r_i = root
s = Laplace variable
T_{ij} = time constant for G_{ij}
T_4 = temperature on 4th tray
T_{17} = temperature on 17th tray
x_D = distillate mass fraction (%)
x_B = bottoms mass fraction (%)
x_i = controlled variable

Greek Letters

α = defined by Eq. 26
ϵ_i = error in loop i
ζ = disturbance variable

ζ^* = disturbance effect defined by Eq. 2
κ = Rynsdorp interaction measure
λ = Bristol-relative gain element
λ_o = steady-state value of λ
Λ_i = $C_i G_{ii}$
π = 3.14159. . .
τ = $\tau_{22} - \tau_{11}$
τ_{ij} = dead time in G_{ij}
ω_{ui} = single-loop ultimate frequency for loop i
ω_{uiI} = ultimate frequency for interacting control of loop i

Indices

i = 1, 2
j = 1, 2

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Estimation of Activity Coefficients in Concentrated Sulfite-Sulfate Solutions

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The electrolyte theory of Pitzer and coworkers is used to estimate geometric mean-ion activity coefficients in strong electrolyte solutions at 25-55°C. By using Pitzer's equations with ion-pair parameters estimated in this paper, one can calculate activity coefficients to moderately high ionic strengths in aqueous mixtures containing the following ions: Na^+, K^+, Mg^{2+}, Ca^{2+}, Cl^-, ClO_3^-, ClO_4^-, HCO_3^-, HSO_3^-, CO_3^{2-}, SO_4^{2-}, SO_3^{2-}, and $S_2O_5^{2-}$. Mixtures containing these ions are present in $CaO/CaSO_4$ flue gas desulfurization units and in laboratory experiments aimed at understanding the chemistry occurring in such units. The computed activity coefficients are relatively insensitive to the values of the estimated quantities. The basis of the estimates, as well as their effect on calculated activity coefficients, is discussed. It is concluded that activity coefficients are estimated with an accuracy of ±25% or better.

SCOPE

Optimizing the performance of $CaO/CaCO_3$ flue gas desulfurization scrubbers requires understanding of the chemistry occurring in these units and knowledge of the thermodynamic properties of the chemical species present. Because some of the chemistry, particularly the sulfur chemistry, is complex, experimental values for the thermodynamic properties of many of the species which may be present are not available and these properties have to be estimated. The present work describes estimates of the activity coefficients at 25°-55°C of the strong electrolytes, Na^+, K^+, Mg^{2+}, Ca^{2+}, Cl^-, ClO_3^-, ClO_4^-,HCO_3^-, HSO_4^-, HSO_3^-, CO_3^{2-}, SO_4^{2-}, SO_3^{2-}, and $S_2O_5^{2-}$, present in

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